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Generalized Hermitian operators in Hilbert space

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GENERALIZED HERMITIAN OPERATORS

IN

HILBERT SPACE

by

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I. INTRODUCTION

The theory of Hilbert space, as initiated by D. Hilbert and later developed mainly by J.v. Neumann, F. Riesz, and M.H. Stone, has been an attractive subject to mathematicians during recent years. The beauty of the theory itself and its great capability of applications to the theories of differential equations, integral equations, quantum mechanics, etc. all hastened its progress. Today it is indeed very fruitful.

Hermitian operators and self-adjoint operators, the latter a special case of the former, are no doubt the real spirit of the theory. They play an important role both theoretically and practically. A Hermitian operator X is characterized by the condition that $X \leq X^*$ while a self-adjoint one by the equality of X and X^*. But it is these simple properties from which interesting results have been derived.

The theory of Hermitian operators is certainly well-developed today. But most of the work which has been done is restricted to the bounded case only. There is also very little work which looks at the Hermitian operator from a more general point of view. In this thesis an attempt was made to get a class of operators which are more general than Hermitian ones yet at the same time enjoy not much less interesting properties. Naturally we prefer not to be confined to the bounded case only, and thus make a contribution to the theory of unbounded operators. It is very likely that a class which is entirely satisfactory does not exist at all. However, a relatively appropriate one would probably be the class in which a member X is characterized by the following conditions:

(1) X^* exists,

(2) $D(X) \cdot D(X^*)$ generates S,

(3) $X = X^*$ over $D(X) \cdot D(X^*)$.

It is obvious that a Hermitian operator is a special case of the above one, and the contraction X' of X over $D(X) \cdot D(X^*)$ is a Hermitian operator. An additional condition $D(X) \cdot D(X^*) = D(X)$ makes X Hermitian.

We will call, just for simplicity, such an X a generalized Hermitian operator, and X', the Hermitian contraction of X.

A few questions naturally arise, for instance:

(1) to what extent does X' determine X?

(2) under what condition will an extension of a Hermitian operator be a generalized Hermitian operator?

(3) if X' is not maximum, how does a maximum extension of X' intersect X?

(4) given a Hermitian operator Z, what are and can we determine all X such that $X' = Z$?

(5) in the closed linear case, is a matrix which

represents X not "far" from being Hermitian?

There are, of course, numerous others.

This thesis is a preliminary study of the class of operators of the above kind. Some of the questions mentioned above are partly answered. But at the present stage, the answers are far from complete.

The thesis contains five chapters. The second chapter is an introduction to the theory of Hilbert space. It serves to make clear the background and offers preliminary knowledge required for later discussions.

The third chapter starts with the consideration of an appropriate generalization of Hermitian operators. It is followed by a discussion of Hermitian contractions of operators in a way similar to that of adjoints of operators. We found that sometimes "'" and "*" act similarly.

In Chapter IV, questions (1), (2) and (4) are partly answered. It seems that X and X' are not very closely related as far as elementary properties are concerned. But in the linear bounded case we found that X and X' must be equal. This shows that the generalization can be proper only for unbounded operators. No satisfactory results have been obtained for questions (2) and (4). One of the difficulties, as will be pointed out later, comes when we begin to deal with elements which are outside of the domain of the adjoint of a Hermitian operator. Another difficulty concerns the interrelations between elements which are in the domain of the adjoint but not in the domain of the Hermitian operator. Question (3) is excluded from the present paper.

In Chapter V a discussion of question (5) is given. We point out a few pathological characteristics in the matrix representation of closed linear proper generalized Hermitian operators. The other half of Chapter V contains several sets of necessary and sufficient conditions that a quadratic matrix generate a generalized Hermitian operator. The results tend to show that our generalization of Hermitian operators is quite natural.

II. HILBERT SPACE AND OPERATORS IN HILBERT SPACE

A brief sketch of the theory of Hilbert space will be given as an introduction. It includes all definitions and theorems which will be used in the following chapters. We will carefully define all terms so that there will be no ambiguity. All useful theorems will be quoted for easy reference. But proofs are omitted to avoid lengthy discussions. For complete knowledge the readers are referred to works of J.v. Neumann (3,4,5,6,7), M.H. Stone (11), F. Riesz (9,10), F.J. Murray (2), and B.v. Sz. Nagy (12).

A. Abstract Hilbert Space

Hilbert space was first introduced by D. Hilbert (1) in his study of linear integral equations. However, postulational treatment was made by J.v. Neumann (3,4,5) some twenty years later.

<u>Definition 1.1</u>. A set S of elements $f, g, \cdots$ is called a Hilbert space if it satisfies the following five postulates:

<u>Postulate A</u>. S is a linear space over the field of complex numbers.

<u>Postulate B</u>. For every pair f, g of elements of S, a unique complex number (f,g) is defined which satisfies

(1) $\overline{(f,g)} = (g,f)$,

(2) $(f,f) > 0$ if $f \neq 0$,

(3) $(af,g) = a(f,g)$ (where a is a complex number) ,

(4) $(f+g,h) = (f,h) + (g,h)$.

(f,g) is called the inner product of f and g.

Before proceeding to the other three postulates, we give the following definitions.

Definition 1.2. The length $||f||$ of an element f of S is the positive value of $\sqrt{(f,f)}$. The distance between two elements f and g is $||f-g||$.

It can be proved without difficulty that $||f-g||$ defines a topology in S. In fact we will use it as the topology of S. All terms like: "limiting element of a set of elements of S" ; "a subset of S which is everywhere dense in S"; will be defined in terms of that topology.

Postulate C. For any positive integer n, there exists a set of n linearly independent elements in S.

Postulate D. S is separable; i.e., there exists a sequence of elements in S, $f_1, f_2, \cdots$, which is everywhere dense in S.

Postulate E. S is complete; i.e., every Cauchy sequence[a] of S has a limiting element in S.

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[a] A sequence (f_n) is called a Cauchy sequence if $||f_m - f_n|| < \varepsilon$ when $m,n > n_o(\varepsilon)$.

A set which satisfies the above postulates is sometimes called an abstract Hilbert space to be distinguished from its realizations. Well-known examples of abstract Hilbert space are: the set of all infinite-tuples of complex numbers and the set of all functions, whose squares are Lebesgue-integrable.

Later S will exclusively be used to denote a Hilbert space; $f, g, h,$ elements of S and $a, b, c,$ complex numbers.

B. Subsets and Operators

<u>Definition 1.3</u>. A subset M of S is called a linear manifold (l.m.) if, whenever f and g belong to M, $af + bg$ also belong to M.

<u>Definition 1.4</u>. A subset M of S is called a closed manifold (c.m.) if every limiting element of M is an element of M.

<u>Definition 1.5</u>. A subset of S is called a closed linear manifold (c.l.m.) if it is closed and linear.

<u>Definition 1.6</u>. The set (M) of all finite linear combinations of M is a l.m. It is called the l.m. generated by M.

<u>Definition 1.7</u>. The set $\langle M \rangle$ consisting of all elements of (M) and their limiting elements is a c.l.m. It is called the c.l.m. generated by M. We also say that $\langle M \rangle$ is generated by M.

<u>Definition 1.8.</u> An operator in S is a single-valued function X defined over some subset D of S in such a way that Xf is in S corresponding to each element f of D. The set D is called the definition domain, $D(x)$, of X. The set of all Xf where $f \varepsilon D(x)$ is called the range, $R(x)$, of X.

<u>Definition 1.9.</u> An operator X is called linear (a l.o.) if, whenever $f, g \varepsilon D(x)$, $af + bg \varepsilon D(x)$ and $X(af + bg) = aXf + bXg$.

<u>Definition 1.10.</u> An operator X is called closed (a c.o.) if, for any sequence (f_n) of elements of $D(x)$, the first two of the following conditions imply the third:

(1) $f_n \rightarrow f$,

(2) $x f_n \rightarrow g$,

(3) $f \varepsilon D(x)$ and $xf = g$.

<u>Definition 1.11.</u> An operator is called closed linear (a c.l.o.) if it is closed and linear.

<u>Definition 1.12.</u> An operator X is called bounded (a b.o.) if there exists a constant C such that $||xf|| \leq C||f||$ for all $f \varepsilon D(x)$.

<u>Definition 1.13.</u> An operator X is called a $*$-operator $(* - 0)$ if $D(x)$ generates S.

<u>Definition 1.14.</u> An operator X is called symmetric (a s.o.) if $(xf, g) = (f, xg)$ for all $f, g \varepsilon D(x)$.

<u>Definition 1.15</u>. An operator is called Hermitian (a H. o.) if it is a symmetric $*$-operator.

<u>Definition 1.16</u>. An operator U is called unitary (a u.o.) if it is closed linear, everywhere defined[a] and $||U f|| = ||f||$ for all f.

<u>Definition 1.17</u>. An operator Y is called an extension of X if $D(X) \leqq D(Y)$ and $Yf = Xf$ for every $f \varepsilon D(X)$. We write $X \leqq Y$. If $X \leqq Y$ and there exists at least one element of $D(Y)$ which is not in $D(X)$, then Y is called a proper extension of X. We write $X < Y$. If $X \leqq Y$ $(X < Y)$, then X is called a contraction (proper contraction) of Y.

<u>Definition 1.18</u>. Y is called a linear (closed linear) extension of X if $X \leq Y$ and Y is linear (closed linear). We write that Y is a l.e. (c.l.e.) of X.

<u>Theorem 1.1</u>. If X has a l.e. (c.l.e.), then X has a least l.e. (c.l.e.)(denoted by $\hat{X}$ $(\tilde{X})$) in the sense that:

(1) $\hat{X}$ $(\tilde{X})$ is a l.e. (c.l.e.) of X,

(2) every l.e. (c.l.e.) of X is an extension of $\hat{X}$ $(\tilde{X})$[b].

<u>Definition 1.19</u>. If X is a $* - o$ and if f is an element of S for which there exists another elment f^* in S such that $(f, Xg) = (f^*, g)$ for all $g \varepsilon D(X)$, then f^* is uniquely determined by f[c]. The operator defined by

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[a]X is said to be everywhere defined if $D(X) = S$.

[b]Reference (11), p. 45, Th. 2.10.

[c]Reference (11), p. 41, Th. 2.6; (4), p. 62, Def. 13.13 and Th. 13.14.

$X^*f = f^*$ for all such f is called the adjoint of X.

Theorem 1.2. The adjoint of a $*$ - o is always c.l.[a]

Theorem 1.3. If X is a $*$ - o, and $\hat{X}$, $\tilde{X}$ exist, then $X^* = \hat{X}^* = \tilde{X}^*$.[b]

Theorem 1.4. If X is a $*$ -o, then X^* is a $*$ - o, if and only if $\tilde{X}$ exists. In this case $X^{**} = \tilde{X}$.[c]

Theorem 1.5. Let X and Y be two $*$ - o, then the following inequalities hold whenever they are defined.

$$X \leq Y \text{ implies } X^* \geq Y^* ,$$

$$X^* + Y^* \leq (X + Y)^* ,$$

$$X^* Y^* \leq (YX)^* ,$$

$$(aX)^* \geq \bar{a} X^* .$$ [d]

Theorem 1.6. If X is a b.l. $*$ - o, then $\tilde{X}$, X^* exist, and are every-where defined and bounded.[e]

C. Hermitian, Maximum Hermitian, and Self-Adjoint Operators.

Hermitian, maximum Hermitian, and self-adjoint operators

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[a] References (11), p. 43, Th. 2.8; (4), p. 62.
[b] References (4), p. 62; (6) p. 301, Satz 2.
[c] References (4), p. 63, Cor. 2; (6), p. 301, Satz 2.
[d] Reference (11), p. 43.
[e] Reference (4), p. 66, Th. 13.21.

are the most interesting and important ones among all operators in Hilbert space. We will give a summary of some of the known results which will be useful in later discussions. For further references see listed works of J.v. Neumann and M.H. Stone.[a]

H.o. was defined in a previous section (Def. 1.15). The following theorem gives an alternative definition.

<u>Theorem 1.7</u>. $A^* - o$ X is Hermitian if and only if $X \leq X^*$.[b]

<u>Definition 1.20</u>. A H.o. X is said to be maximum (a max. H.o.) if there exists no H.o. which is a proper extension of X.

<u>Definition 1.21</u>. A H.o. X is said to be self-adjoint (a s.a.o.) if $X = X^*$.

<u>Definition 1.22</u>. A H.o. X is said to be essentially self-adjoint (a e.s.a.o.) if X^* is s.a.

<u>Theorem 1.8</u>. If X is Hermitian, then $\hat{X}$ and $\tilde{X}$ exist. They are also Hermitian.[c]

<u>Theorem 1.9</u>. Max. H.o. and s.a.o. are c.l.[d]

<u>Theorem 1.10</u>. If X is Hermitian, then X is e.s.a. if and only if X^* is Hermitian.

Proof: If X and X^* are Hermitian, then $X \leq \tilde{X} = X^{**} \leq X^*$

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[a] References (3), (4), (5), (7), (11).

[b] References (11), p. 49; (4), p. 63, Th. 13.16.

[c] References (3), p. 70, Satz 9; (11), p. 49, Th. 2.12.

[d] References (3), p. 72; (11), p. 51, Th. 2.14.

and $X^* \leq X^{**}$, so $X^* = X^{**}$. If X is e.s.a., then X^* is s.a., so it is Hermitian.

<u>Theorem 1.11</u>. Every b.l.s.o. is e.s.a. A b.l.s.o. is s.a. if and only if it is everywhere defined.[a]

<u>Definition 1.23</u>. If X is a H.o., then every element of $D(X^*)$ is called an extended element (e.e.) of X. They are classified into three classes: positive (+ e.e.), negative (- e.e.), and zero (o e.e.) according as $I(X^*f,f) \gtreqless 0$, (where $I(X^*f,f)$ is the imaginary part of (X^*f,f)).

<u>Theorem 1.12</u>. A H.o. X is max. if and only if all its o.e.e. are contained in $D(X)$. It is s.a. if and only if all its e.e. are contained in $D(x)$.[b]

D. Bases in Hilbert Space

<u>Definition 1.24</u>. Two elements f,g of S are said to be orthogonal if $(f,g) = 0$. f is called normalized if $||f|| = 1$. A set M of elements of S is called orthonormal if each element of M is normalized and each pair of elements of M is orthogonal.

<u>Definition 1.25</u>. By orthonormalization of a sequence (f_n) of elements of S is meant the performance of the following procedure for that sequence:

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[a] Reference (11), p. 58, Th. 2.24.

[b] Reference (3), p. 72, Satz 11 and Def. 9.

(1) deleting all members of (f_n) which are zero and then all members which can be expressed as linear combinations of preceding members,

(2) applying Schmidt's process[a] to the remaining members of the sequence.

Theorem 1.13. The sequence obtained by orthonormalization of a given sequence is orthonormal. They generate the same c.l.m.[b]

Theorem 1.14. S contains an orthonormal set with a denumerable number of elements which generates S. Any orthonormal set which generates S contains a denumerable number of elements.[c]

Definition 1.26. An orthonormal sequence of elements which generates S is called a complete orthonormal sequence (c.o. N.S.) of S.

By Th. 1.14, S contains a c.o.N.S.

Theorem 1.15. If (ϕ_n) is a c.o.N.S. in S, then $\sum_{n=1}^{\infty} a_n \phi_n$ (a_n are complex numbers) is convergent if and only if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.[d]

Theorem 1.16. Let (ϕ_n) be a c.o.N.S. in S, then,

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[a] References (3), p. 69; (4), p. 16, Th. 12.13; (11) p. 13.

[b] References (3), p. 69; Satz. 8; (11), p. 12, Th. 1.13.

[c] References (3), p. 69, Satz. 8; (4), p. 23, Th. 12.18, 12.19; (11), p. 13, Th. 1.14.

[d] References (3), p. 67, Satz. 5; (4), p. 20, Th. 12.16.

for every element f of S, the series $\sum_{n=1}^{\infty}(f,\phi_n)\phi_n$ is convergent and its sum is f. In other words, every f of S can be expressed as $\sum_{n=1}^{\infty} a_n \phi_n$ where $a_n = (f, \phi_n)$, $n = 1,2,\cdots$.[a]

Theorem 1.17. If (ϕ_n) is a c.o.N.S. in S and f,g are two elements, then

$$(f,g) = \sum_{n=1}^{\infty} (f,\phi_n)(\phi_n,g).$$ [b]

E. Matrix Representations of Closed Linear Operators

It can be shown that closed linear operators are representable by infinite matrices in a certain manner. There are some theorems proved by J.v. Neumann and M.H. Stone which are fundamental. They will be quoted in the following without proofs.

We will begin by definitions of some special infinite matrices.

Definition 1.27. An infinite matrix A is called Hermitian (a.H.m.) if $A = A^*$, where A^* is the adjoint of A.[c]

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[a] References (3), p. 68, Satz. 7; (11), p. 10, Th. 1.9.

[b] Same as above.

[c] $B = (b_{ij})$ is the adjoint of $A = (a_{ij})$ if $b_{ij} = \overline{a_{ji}}$, $i,j = 1,2,\cdots$.

<u>Definition 1.28</u>. An infinite matrix $A = (a_{ij})$ is called quadratic (a q.m.) if $\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$ for all $i = 1,2,\cdots$.

<u>Definition 1.29</u>. An infinite matrix $u = (u_{ij})$ is called unitary (u.m.) if

$$\sum_{k=1}^{\infty} U_{ik}\,\overline{U}_{jk} = \begin{matrix} 1 & \text{for } i = j\,, \\ 0 & \text{for } i \neq j\,. \end{matrix}$$

$$\sum_{k=1}^{\infty} U_{ki}\,\overline{U}_{kj} = \begin{matrix} 1 & \text{for } i = j\,, \\ 0 & \text{for } i \neq j\,. \end{matrix}$$

<u>Theorem 1.18</u>. If (ϕ_n) and (ψ_n) are two c.o.N.S. in S, then the matrix $U = U_{ij})$, where $U_{ij} = (\phi_i, \psi_j)$, $i,j = 1,2,\cdots$, is unitary.

Proof. Use Th. 1.17.

<u>Theorem 1.19</u>. If X is a c.l. H.o., then there exists a c.o.N.S. (ϕ_n) in $D(X)$ such that $X = \widetilde{X}_1$, where X_1 is the contraction of X over (ϕ_n). Define $a_{ij} = (X\,\phi_i,\ \phi_j)$, $i,j = 1,2,\cdots$, then $A = (a_{ij})$ is a H.q.m. If (ψ_n) is another c.o.N.S. in $D(X)$, and $B = (b_{ij})$ is the corresponding H.q.m. then there exists a unitary matrix $U = (u_{ij})$ such that

$$b_{ij} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{im}\, a_{mn}\, \overline{U}_{jn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} U_{im}\, a_{mn}\, \overline{U}_{jn}\,, \quad i,j = 1,2,\cdots\,.$$

Or $B = U(AU^*) = (UA)U^*$.

<u>Theorem 1.20</u>. If X is a c.l. $*$-o., then there exists a c.o.N.S. (ϕ_n) in $D(X)$ such that $X = \widetilde{X}_1$ where X_1 is the contraction of X over (ϕ_n). Define $a_{ij} = (X\phi_i, \phi_j)$, $i, j = 1, 2, \cdots$, then $A = (a_{ij})$ is a q.m. Further, any member, say ϕ_i, of the sequence (ϕ_n) is in $D(X^*)$ if and only if $\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$, and $X\phi_i = X^*\phi_i$ if and only if $a_{ij} = \overline{a}_{ji}$, $j = 1, 2, \cdots$.

<u>Definition 1.30</u>. Following the notations of the above theorems, (ϕ_n) is called a determining set (d.s.) of X. X_1 is called the elementary operator (e.o.) of X for the d.s. (ϕ_n). Matrix A is said to belong to the operator X for (ϕ_n).

The converses of the above theorems are:

<u>Theorem 1.21</u>. Let $A = (a_{ij})$ be a H.q.m. Then, for any c.o.N.S. (ϕ_n), there exists one and only one c.l.H.o. X to which A belongs for (ϕ_n).

<u>Theorem 1.22</u>. Let $A = (a_{ij})$ be a q.m. Then, for any c.o.N.S. (ϕ_n), there exists one and only one c.l. $*$-o. X to which A belongs for (ϕ_n). Further, the statements about ϕ_i in Th. 1.20 remain valid here.

<u>Definition 1.31</u>. The operator X in the above theorems is said to be generated by the matrix A by the c.o.N.S. (ϕ_n).

III. HERMITIAN CONTRACTIONS OF OPERATORS IN HILBERT SPACE

A. Hermitian Operators and Their Generalization

As is stated in Th. 1.7, a H.o. X is characterized by the condition that $X \leq X^*$. This, if explained in a more comprehensive way, means that an operator X is Hermitian if and only if it satisfies the following three conditions (we prefer to state them separately because they are independent of each other):

(1) X is a $*$-operator, so that X^* exists;

(2) $D(X) \cdot D(X^*) = D(X)$ generates S;[a]

(3) $X = X^*$ over $D(X) \cdot D(X^*)$.

We are interested in classes of operators which are in natural ways more general than H.o. But at the same time we do not wish to generalize too much. So, before choosing a class of operators which will be appropriate, at least subjectively, for our purpose, an analysis of the above three conditions is necessary.

In order to take advantage of the theory of adjoints, we assume that X, the operator under consideration, is a

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[a] If A and B are two sets, then $A \cdot B$ is the intersection of A and B.

$*$ -operator.

As for conditions (2) and (3), there are possibilities other than those we have stated above. For instance, four different cases can be obtained by combinations from the following two sets of conditions.

I
(1) $D(X) \cdot D(X^*)$ generates S ,
(2) $D(X) \cdot D(X^*)$ does not generate S.

II
(3) $X = X^*$ over $D(X) \cdot D(X^*)$,
(4) $X \neq X^*$ over $D(X) \cdot D(X^*)$.

A few examples will be given for illustration.

Example 1. Let $Xf = if$ for all $f \varepsilon S$. The necessary and sufficient condition that $g \varepsilon D(X^*)$ is that there exists $g^* \varepsilon S$ such that $(g, Xf) = (g^*, f)$ for all $f \varepsilon D(X)$. Since $Xf = if$, we see that $g^* = -ig$. This means: X^* is defined for all $g \varepsilon S$ and $X^*g = -ig$. In this case condition (1) is fulfilled while (3) is not, for $Xf = X^*f$ only if $f = 0$.

Example 2. Let (ϕ_n) be a c.o.N.S. and $A = (a_{ij})$ be a q.m. of the following form:

$$\begin{matrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

Define $X\phi_i = \sum_{j=1}^{\infty} a_{ij}\phi_j$ for $i = 1,2,\cdots$. Since none of $\sum_{i=1}^{\infty} |a_{ij}|^2$, $j = 2,3,\cdots$. converges, by Th. 1.22, ϕ_2, ϕ_3, $\cdots$, are not in $D(X^*)$. But $\phi_1 \varepsilon D(X^*)$ and $X\phi_1 = X^*\phi_1 = \phi_1$. So in this case, condition (3) is fulfilled while (1) is not.

The operators shown in the above examples are "quite" non-Hermitian. So it seems to us that if an operator does not satisfy conditions (1) and (3), then it is very unlikely that it will share any of the interesting properties of Hermitian operators. On the other hand, if an operator X does satisfy the two conditions, then it is "not far" from being Hermitian, because the additional condition that $D(X) \cdot D(X^*) = D(X)$ insures its Hermitianness. Therefore we suggest that the class of operators which satisfy conditions (1) and (3) is an appropriate generalization of the class of all Hermitian operators. However, for use as a definition, we prefer the following slightly more general statement:

<u>Definition 2.1</u>. If X is a $*$-o. such that the set of all f for which $Xf = X^*f$ generates S, then X is called a generalized Hermitian operator (g.H.o.). The contraction X'[a] of X over that set is a H.o. It is called the Hermitian contraction of X.

The following few lemmas are obvious.

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[a] X' will always be used to denote the Hermitian contraction of a g.H.o. X.

<u>Lemma 2.1</u>. X is a g.H.o. if and only if there exists a H.o. Z such that $Z \leq X$ and $Z \leq X^*$. In this case $X' \geq Z$.

<u>Lemma 2.2.</u> If X is a g.H.o., then $D(X')$ is the set of all f such that $(f,Xg) = (Xf,g)$ for all $g \varepsilon D(X)$.

<u>Lemma 2.3</u>. If X is a g.H.o., then $X' \leq X \leq X'^*$.

Proof. By Def. 2.1, $X' \leq X$ and $X' \leq X^*$. Also $X'^* \geq X^{**} = \tilde{X} \geq X$ (Th. 1.5).

<u>Lemma 2.4</u>. If X is a g.H.o., then $X = X^*$ over $D(X) \cdot D(X^*)$. So X' is the contraction of X over $D(X) \cdot D(X^*)$.

Proof. X and X^* have a common extension X'^*. Hence $X = X^*$ over $D(X) \cdot D(X^*)$.

<u>Lemma 2.5</u>. X is a g.H.o. if and only if there exists a H.o. Z such that $Z \leq X \leq Z^*$. In this case $X' \geq Z$.

Proof. By Lemma 2.3, we see that the condition is necessary. On the other hand, if $Z \leq X \leq Z^*$, then $X \geq Z$ and $X^* \geq Z^{**} = \tilde{Z} \geq Z$. By Lemma 2.1, X' exists and $X' \geq Z$.

It is important to show that there do exist generalized Hermitian operators which are not Hermitian. Otherwise the generalization will be trivial.

Example. Let X be a max. H.o. which is not s.a.[a] Then there exists an f in S which is an e.e. of X, but not of the o-class. Let Y be an operator defined in

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[a] See Reference (3), p. 98, Satz. 37.

such a way that

$$Y\, g = X\, g \quad \text{for all} \quad g \in D(X)\ ,$$
$$Y\, f = X^*\, f \qquad .$$

It is easy to show that Y is a g.H.o., but not Hermitian.

We will consider in the following chapters some problems concerning the generalized Hermitian operators. Results obtained are divided into three parts. The present chapter is a study of the relations between the Hermitian contractions, the adjoints, the least closed linear extensions, etc. of generalized Hermitian operators.

B. Hermitian Contractions of X, $\hat{X}$, $\tilde{X}$ and X^*

<u>Theorem 2.1</u>. If X is a g.H.o., then $\hat{X}$, $\tilde{X}$, and X^* exist. They are also g.H.o., and $X' \leq \hat{X}' \leq \tilde{X}' = X^{*'}$.

Proof. By Def. 2.1, if X is a g.H.o., then X^* exists and $D(X^*) \geq D(X')$ which generates S. Hence $\tilde{X}$ and $\hat{X}$ also exist (Th. 1.4). Further $X' \leq \hat{X} \leq \tilde{X} = X^{**}$, and $X' \leq X^* = \hat{X}^* = \tilde{X}^*$. So by Lemma 2.1, X', $\tilde{X}'$ and $X^{*'}$ exist.

It is obvious that $X' \leq \hat{X}' \leq \tilde{X}'$. It is also easy to show that $\tilde{X}' = X^{*'}$, because $f \in D(\tilde{X}')$ if and only if $\tilde{X}\, f = \tilde{X}^* f (= X^* f)$ while $f \in D(X^{*'})$ if and only if $X^* f = X^{**} f (= \tilde{X}\, f)$.

<u>Remark 1</u>. If $\tilde{X}$ is a g.H.o., it is not necessary that X is also such an operator. Consider $\tilde{Y}$ where Y is the operator given by the example on p. 20. Let Y_1 be the contraction of $\tilde{Y}$ over a d.s. of $\tilde{Y}$. Then $\tilde{Y}_1 = Y$, but Y_1 is not a g.H.o.

<u>Remark 2</u>. There are examples which show that $\tilde{X}' > X'$ and $\hat{X}' > X'$. Let Y be a l.g.H.o. (e.g. any l.e. of the operator given by the example on p. 20.). It will be shown later that if any g.H.o. Y is (closed) linear, then Y' is also (closed) linear.[a] Let Y_0 and Y_1 be the contractions of Y and Y' over $D(Y) - (0)$ and $D(Y') - (0)$ respectively.[b] We see that $\hat{Y}_0 = Y$. It will now be shown that $Y_0' = Y_1 < Y' = \tilde{Y}_0'$.

Since $Y_1 \leq Y_0 \leq Y \leq Y'^* \leq Y_1^*$, $Y_0' \geq Y_1$ (Lemma 2.5). But $Y_0 \leq Y$ and $Y_0^* = Y^*$, hence $Y_0' \leq Y'$. As $D(Y_0')$ does not contain 0, $Y_0' \leq Y_1$. Therefore $Y_0' = Y_1$.

<u>Theorem 2.2</u>. If X is a g.H.o., then $\tilde{X}' \geq X'^{\sim}$ and $\hat{X}' \geq X'^{\wedge}$.

Proof. Since $\tilde{X}' \geq X'$, $(X')^{\sim} \geq X'$. So $\tilde{X}' \geq X'^{\sim}$.[c] Similarly we show $\hat{X}' \geq X'^{\wedge}$.

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[a] See Th. 3.1.

[b] If N is a subset of M, then $M-N$ is the set of all elements of M which are not in N.

[c] See Th. 3.1.

Remark. If a g.H.o. is closed and linear, then in the above the two equalities hold. However, closure and linearity are not necessary for equality. The operator Y given by the example on p. 20 is not closed and not linear. But it can be shown that $Y' = \hat{Y}' = \tilde{Y}' = X$ (where X is the max. H.o. given there).[a] So $Y'^{\wedge} = \hat{Y}'$ and $Y'^{\sim} = \tilde{Y}'$.

Theorem 2.3. If X is a g.H.o., then $\tilde{X}' = X'^{\sim}$ and $\hat{X}' = X'^{\wedge}$ if and only if

$$(D(\tilde{X}) - D(X')) \cdot (D(X^*) - D(X'^{\sim})) = 0 ,$$

$$(D(\hat{X}) - D(X')) \cdot (D(X^*) - D(X'^{\sim})) = 0 ,^{b}$$

respectively.

Proof. If there exists an $f \varepsilon (D(\tilde{X}) - D(X')) \cdot (D(X^*) - D(X'^{\sim}))$, then $f \varepsilon D(\tilde{X}) \cdot D(X^*)$, but $f D(X')$. If $f \varepsilon D(\tilde{X}) \cdot D(X^*)$, then $\tilde{X} f = X^* f$, so $f \varepsilon D(\tilde{X}')$ (Lemma 2.4). Hence $\tilde{X}' > X'^{\sim}$. Thus the condition is necessary.

On the other hand if $\tilde{X}' > X'^{\sim}$, then there exists a f in $D(\tilde{X}) \cdot D(X^*)$ which is not in $D(X'^{\sim})$. So f is in $D(\tilde{X})$ and $D(X^*)$, but f is certainly not in $D(X')$. This shows that f is in $(D(\tilde{X}) - D(X')) \cdot (D(X^*) - D(X'^{\sim}))$. Hence the condition is sufficient.

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[a] If X is a max. H.o. and Y is such that $X \leq Y \leq X^*$, then $Y' = X$, see Th. 3.6.

[b] This means that the intersection of the two sets is null.

<u>Theorem 2.4</u>. If X and Y are two g.H.o., then the following relations hold whenever they are defined:

$$X' + Y' \leq (X + Y)' ,$$

$$(a X)' \geq \bar{a} X' , \text{ (where } a \text{ is a constant).}$$

Proof. They can easily be verified by means of Th. 1.5.

We like to point out that for X' and Y', no assertion like the first one of Th. 1.5 can be made. On the contrary, whether $X' \geq Y'$ is entirely independent of whether $X \leq Y$. The following examples serve to illustrate the situation.

<u>Example 1</u>. Let X and Y be the operators given by the example on p.20 . If $\tilde{X}_1 = X$ and Y_1 is the contraction of Y over $D(X_1) + (f)$ where f is that e.e.[a], then $Y_1 < Y$ while $Y_1' = X_1 < X = Y'$.

<u>Example 2</u>. Let X be a c.l.H.o. with deficiency indices m and n where $m \neq n \neq 0$.[b] Then there exists a max. extension X_m of X such that $X_m > X$. Let f be a non-c-class e.e. of X, and define $Y f = X^* f$ and $Y g = X_m g$ for all $g \in D(X_m)$ then we see that $Y < X^*$, but $Y' = X_m > X = X^{*'}$.

C. Hermitian Contractions of X^{-1} and X^*

<u>Theorem 2.5</u>. If X is a g.H.o. such that:

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[a] If A and B are two sets, then $A + B$ is set of all elements in A and B.

[b] For the existence of such an operator see reference (3), p. 101.

(1) it possesses an inverse X^{-1},[a]

(2) $R(X')$ generates S,

then X^{-1} is a g.H.o. and $(X^{-1})' = (X')^{-1}$.

Proof. Under the given conditions, it can be proved that $(X^{-1})^*$ exists and $(X^{-1})^* = (X^*)^{-1}$.[b] Now, by hypothesis, $D((X')^{-1}) = R(X')$ generates S. Let A and B denote respectively $D((X')^{-1})$ and the set of all f such that $X^{-1}f = (X^{-1})^*f$. If we can show that $A = B$ and $X^{-1}f = (X')^{-1}f$ for all $f \varepsilon A$, then $(X^{-1})'$ exists and $(X^{-1})' = (X')^{-1}$.

Let $f \varepsilon A$, then $f = X'g$ for some $g \varepsilon D(X')$ and $(X')^{-1}f = g$. Since $X'g = Xg = X^*g$, $X^{-1}f = (X)^{-1}f = (X^{-1})^*f = g$. So $f \varepsilon B$. This shows that $A \leq B$. Likewise, we show that $B \leq A$. Hence $B = A$ and generates S. It follows that $(X^{-1})'$ exists and $(X^{-1})' = (X')^{-1}$.

The last topic which we will discuss in this chapter is the square of a generalized Hermitian operator. The following question is often asked: If it is known that X is an operator of certain class, when does X^2 belong to the same class. For example, it is known that if X is a s.a.o. or a max. H.o., then X^2 is also Hermitian. M. Neumark (8) showed that this is not true for H.o. in general. He gives an

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[a] For definition of inverse see reference (11), p. 37, Th. 2.2.

[b] Reference (11), p. 43, Th. 2.7.

interesting example. He also found some necessary and sufficient conditions in order that $X^\#$ be Hermitian.

Theorem 2.6. If X and $X^\#$ are both g.H.o., then $(X')^\# \leq (X^\#)'$.

Proof. Use Th. 1.5.

Theorem 2.7. If X is a g.H.o., then the following two conditions are equivalent:

(1) $X^\#$ is a g.H.o., $D(X') \leq D((X^\#)')$, and $D(X') \leq D((X^*)^\#)$,

(2) $D(X')$ is invariant under X.[a]

Proof. If (2) is fulfilled, then for every $f \varepsilon D(X')$, $X^\# f = (X^*)^\# f$. So $D(X^\#) \geq D(X')$ and $D((X^*)^\#) \geq D(X')$. Hence $(X^\#)^*$ exists and $(X^\#) f = (X^*)^\# f = X^\# f$ for $f \varepsilon D(X')$. It follows that $(X^\#)'$ exists and $D(X') \leq D((X^\#)')$. On the other hand if (1) holds, then for every $f \varepsilon D(X')$ we have $X^\# f = (X^\#)^* f = (X^*)^\# f = X^* X f$, so $X f \varepsilon D(X')$. Hence $D(X')$ is invariant under X.

[a] A subset A of $D(X)$ is said to be invariant under X if $X f \varepsilon A$ for all $f \varepsilon A$.

IV. GENERALIZED HERMITIAN OPERATORS AND THEIR HERMITIAN CONTRACTIONS

A. A Few Relationships

Since the H.c. of a g.H.o. X is one of its contractions over a "relatively large" subset of $D(X)$, it would seem that X and X' are closely related with regard to properties like linearity etc. However, the facts are not what we expect, although an important relation--the equivalence of the two operators in the bounded case, can be proved.

Theorem 3.1. If X is a g.H.o., then X' is (closed) linear if X is so. The converse is not true.

Proof. We will give a proof, for illustration, in the case where X is closed. If $f_n \longrightarrow f$ and $X' f_n \longrightarrow g$, then $X f_n = X^* f_n \longrightarrow g$. Since X and X^* are both closed, $Xf = X^*f = g$. Hence $X'f = g$ which shows that X' is closed.

The converse is not true (see the example on page 20). However, $D(X)$ and $D(X')$ always contain the 0 element simultaneously.

Theorem 3.2. If X is a l.g.H.o., then X is bounded if and only if X' is so. In this case, $X = X'$ is an e.s. a.o.

Proof. If X is bounded, then X' must be bounded.

If X' is bounded (it is linear), then by Th. 1.6, 1.10, and 1.11, X'^* is bounded and Hermitian. Since $X'^* \geq X$, X must be bounded and Hermitian. So $X = X'$ is an e.s.a.o.

Remark. The above theorem shows that if a g.H.o. is linear and bounded, then it is no more than Hermitian. Therefore in the study of l.g.H.o., nothing new can be obtained unless we assume both the operator and its H.o. to be unbounded. In other words, a l.g.H.o. X is proper only if X' is an unbounded l.H.o.[a] Therefore the study of l.g.H.o. should be considered as a part of the broader problem of non-symmetric extensions of unbounded linear Hermitian operators.

In the unbounded case, a g.H.o. is in general not affected by its H.o. to a considerable degree.

B. Necessary and Sufficient Conditions

In this section we will give a few criteria insuring that a $*$-o. be a g.H.o.

The following theorem is merely a restatement of Lemma 2.4.

Theorem 3.3. A $*$-o. X is a g.H.o. if and only if $D(X) \cdot D(X^*)$ generates S and $X = X^*$ over that intersection.

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[a] We say that a g.H.o. is proper or it is a proper g.H.o. (p.g.H.o.) if it is a g.H.o. but not Hermitian.

A few questions which arise naturally are: is any extension of a H.o. a g.H.o?; if it is not so, then under what additional conditions will an extension of a H.o. be a g.H.o.? The answer to the first question is in the negative, for we know that if X is a g.H.o., then $\widetilde{X}$ exists, while an arbitrary extension of a H.o. may not even possess a linear extension. As for the second question, so far we know very little about it. A few conditions either necessary or sufficient that an extension of a H.o. be a g.H.o., are, for example:

(1) if X is a g.H.o., it is necessary that $\widetilde{X}$ exists,

(2) if $Z \leq X \leq Z^*$ where Z is Hermitian, then X' exists,

(3) if X is an extension of a s.a.o. Z which possesses $\widetilde{X}$, then X' exists.

However it would be quite difficult to get conditions which are necessary and sufficient for an extension of a H.o. to be a g.H.o. The main difficulty is: if Z is a H.o. and $X > Z$, but $X \not< Z^*$, then we have to deal with elements which are not e.e. of a H.o. Unfortunately there is very little available information about this.

A few simple sufficient conditions will be given for an extension of a H.o. to be a g.H.o.

<u>Theorem 3.4</u>. If X is an extension of a H.o. Z, and

$D(Z) \cdot D(X^*)$ generates S, then X is a g.H.o.

Proof. Since $X^* \leq Z^*$ and $Z \leq Z^*$, $X^* = Z$ over $D(Z) \cdot D(X^*)$. Let Z_1 be the contraction of Z over $D(Z) \cdot D(X^*)$, then Z_1 is a H.o. Now $Z_1 \leq X$, $Z_1 \leq X^*$, Hence X is a g.H.o. (Lemma 2.1).

Theorem 3.5. If X is a c.l. extension of a H.o. Z, $X = Z^*$ over $D(X) \cdot D(Z^*)$, and $X \cup Z^*$ has a c.l. extension, then X is a g.H.o.[a]

Proof. Let $Y = X \cup Z^*$. Since $\tilde{Z} \leq X \leq Y$ and $Z^* \leq \tilde{Y}$, so $Y \geq \tilde{Z} = Z^{**} \geq Y^*$. But $D(Y^*)$ generates S, for $\tilde{Y}$ exists. Hence Y^* is Hermitian. Now

$$Y^* \leq \tilde{Z} \leq X \leq Y \leq \tilde{Y} = (Y^{**}) .$$

By Lemma 2.5, X is a g.H.o.

C. Operators With Given Hermitian Contractions

Theorem 3.6. If Z is a H.o., then X is a g.H.o. and $X' = Z$ if and only if $Z \leq X \leq Z^*$ and no proper Hermitian extension of Z satisfies the same relation.

Proof. If $X' = Z$, then $Z \leq X \leq Z^*$ (Lemma 2.3).

---

[a] Let X and Y be two operators such that $X = Y$ over $D(X) \cdot D(Y)$, then $X \cup Y$ is the operator which coincides with X over $D(X)$ and with Y over $D(Y)$; and $X \cap Y$ is the contraction of Y or X, over $D(X) \cdot D(Y)$.

Suppose there exists a H.o. $Y > Z$ such that $Y \leq X \leq Y^*$, then $X' \geq Y > Z$ (Lemma 2.5). Contradiction! On the other hand, if $Z \leq X \leq Z^*$, then $X' \geq Z$. If $X' = Y > Z$, then $Y \leq X' \leq Y^*$. Contradiction!

Remark 1. If $X' = Z$, then the intersections of $D(X)$ and $D(\hat{Z})$, $D(\tilde{Z})$ each will be no more than $D(Z)$. Otherwise X' will be in each case at least the contraction of X over that intersection which is a proper extension of Z. If Z is e.s.a., then $X' = Z$ if and only if $X = Z$. If Z is a max. H.o., then $X' = Z$ if and only if $Z \leq X \leq Z^*$.

Remark 2. If $Z \leq X \leq Z^*$ and X contains no 0 e.e. of Z other than those in $D(Z)$, then $X' = Z$. Otherwise, if $X' = Y > Z$, let f be in $D(Y)$ but not in $D(Z)$, then $f \varepsilon D(X)$ and is a 0.e.e. of Z.

Theorem 3.7. Given a H.o. Z, then there exists a p.g.H.o. X such that $X' = Z$, if and only if Z is not an e.s.a.

Proof. The necessity was pointed out in Remark 1 in the above. To show its sufficiency, let Z be a H.o. but not e.s.a., then Z^* must be non-Hermitian (Th. 1.10). Hence there exists at least one element f in $D(Z^*)$ such that $(f, Z g) \neq (Z f, g)$ for at least one g in $D(Z^*)$. Let Y be the contraction of X over the set $D(Z) + (f) + (g)$, then $Y > Z$ and $Y' = Z$.

Theorem 3.8. If X is a g.H.o. such that $X' = Z$, then

$$(X \cup \hat{Z}) = \hat{Z} \,,$$

$$(X \cup \tilde{Z})' = (X \cup X^*)' = \tilde{Z}.$$

If Y is another g.H.o. with $Y' = Z$, then $(X \cup Y)' = Z$.

Proof. The theorem can be established by Th. 3.6. For instance: $\hat{Z}^* = Z^* \geq X \cup \hat{Z} \geq \hat{Z}$, hence $(X \cup \hat{Z})' \geq \hat{Z}$. Now, if $(X \cup \hat{Z})' = Z_1 > Z$, then $X \cap Z_1 > Z$ and $X \cap Z_1 \leq X \leq Z_1^* \leq (X \cap Z_1)^*$. So $X' \geq X \quad Z_1 > Z$. Contradiction!

V. MATRIX REPRESENTATIONS OF CLOSED LINEAR GENERALIZED HERMITIAN OPERATORS

A concrete way to study operators is by the use of orthonormal bases and matrices. In finite dimensional Euclidean space, matrices and linear operators are so closely related that they are usually considered as merely two different interpretations of the same thing. For linear bounded operators in Hilbert space, matrices have also been used very successfully; in fact, most of the early developments were based on matrix representations with respect to orthonormal bases (13). The situation, however, is entirely different for unbounded operators. J.v. Neumann (7) obtained a number of what he called pathological results which showed the difficulty in handling unbounded operators by means of matrices. Nevertheless it is still important as well as interesting to study the behavior of matrices of unbounded operators, although in most cases it is complicated. We will, in this chapter, consider this problem for generalized Hermitian operators. But, in line with convention, we will confine ourselves to the closed linear case only, because this is the path of least resistance!

A. Pathological Behavior

We know that a c.l. * -o. can be represented by infinite quadratic matrices in the sense of Th. 1.20. If the operator is Hermitian, then the representing matrices are Hermitian. As generalized Hermitian operators are to some extent similar to Hermitian operators, it might be expected that their representing matrices should correspondingly be closely related to Hermitian matrices. Unfortunately this is far from being the case. We will show that the "Hermitian property" of matrices which represent a closed, linear, generalized Hermitian operator depends entirely on the choice of determining sets. For some choices of d.s. of the operator, we can obtain a matrix which is "almost" Hermitian. But for other choice of a d.s., the result can be quite different.

The following lemma is useful.

<u>Lemma 4.1</u>. If X is a c.l.o. and (ϕ_n) is a d.s. of X, then, for any sequence (f_n) of elements of $D(X)$, the c.o.n.s. (ψ_n), obtained from orthonormalization of a single sequence (g_n) consisting of (ϕ_n) and (f_n), is also a d.s. of X.

Proof. Let the contradictions of X over (ϕ_n), (g_n), and (ψ_n) be X_1, X_2, and X_3. Since $(\phi_n) \leqq (g_n) \leqq D(X)$, $X_1 \leqq X_2 \leqq X$. So $X = \tilde{X}_1 \leqq \tilde{X}_2 \leqq X$. But $\tilde{X}_3 = \tilde{X}_2$. Hence $\tilde{X}_3 = X$ and (ψ_n) is a d.s. of X.

<u>Theorem 4.1</u>. If X is a c.l.g.H.o., then for any given integer n, there exists a d.s. (ϕ_n) of X such that at least n members of (ϕ_n) are in $D(X')$. In case X is proper, then for any n, there exists a d.s. (ϕ_n) of X such that at least n members of (ϕ_n) are not in $D(X')$.

Proof. Let (ϕ_n), (ψ_n) be two d.s. of X and X', respectively. Consider the sequence.

$$\psi_1 , \psi_2 , \cdots, \psi_n, \phi_1, \phi_2, \cdots .$$

By Lemma 4.1 the c.o.N.s. obtained by orthonormalization of the above sequence is a d.s. of X. From the orthonormalization process, we know that the first n members of the new d.s. are $\psi_1, \cdots, \psi_n$. They are in $D(X')$.

Now, using the same notation, if X is not Hermitian, then at least one member, say ϕ_1, of (ϕ_n) is not in $D(X')$. We say that $(\phi_1, \psi_i) \neq 0$ for at least an infinite number of i. Otherwise there are only a finite number of non-vanishing coefficients in the series $\phi_1 = \sum_{n=1}^{\infty} a_n \psi_n$, so ϕ_1 is a finite linear combination of ψ_i whichare in $D(X')$. Contradiction!

Without loss of generality, let us assume that $(\phi_1,\psi_i)\neq 0$ for $i = 1,2,\cdots,n-1$. Consider the sequence,

$$\phi_1, \psi_1, \psi_2, \cdots, \psi_{n-1}, \phi_2, \phi_3, \cdots .$$

For the sake of uniformity, we will relabel it as

$$f_1,\ f_2,\ \cdots,\ f_n,\ \cdots\ ,$$

then $\quad (f_i,\ f_i) = 1,$ for $i = 1,2,\cdots,n$,

$$(f_1,\ f_i) \neq 0, \text{ for } i = 2,\cdots,n\ ,$$

$$(f_i,\ f_j) = 0, \text{ for } i \neq j = 2,\cdots,n\ .$$

By Lemma 4.1, the o.o.n.s. (ϕ_n) obtained by orthonormalization of (f_n) is a d.s. of X. We will show that ϕ_i, $i = 1,\cdots,n$, are not in $D(X')$. Let

$$\phi_i = \sum_{j=1}^{i} t_{ij}\, f_j \quad \text{for } i = 1,\cdots,n\ ,$$

$$f_i = \sum_{j=1}^{i} s_{ij}\, \phi_j \quad \text{for } i = 1,\cdots,n\ ,$$

then $t_{ii} \neq 0$, $s_{ii} \neq 0$, and $t_{ii} = \frac{1}{s_{ii}}$, $i = 1,\cdots, n$. We know that $t_{11} = 1$. It will now be shown that $t_{i1} \neq 0$, $i = 2,\cdots,n$.

$$||\phi_i - t_{i1}\, f_1||^2 = ||\phi_i - t_{i1}\, \phi_1||^2$$

$$= ||\sum_{j=2}^{i} t_{ij}\, f_j||^2, \quad i = 2,\cdots,n\ .$$

So $1 + |t_{i1}|^2 = \sum_{j=2}^{i} |t_{ij}|^2$, or $|t_{i1}|^2 = \sum_{j=2}^{i} |t_{ij}|^2 - 1$.

But $1 = |f_i|^2 = ||\sum_{j=1}^{i} s_{ij}\, \phi_j||^2 = \sum_{j=1}^{i} |s_{ij}|^2$. Hence

$|s_{ij}| \leq 1$, $j = 1,\cdots,i$. So $|t_{ii}| \geq 1$. Now, if $t_{ii} = 0$, then $t_{ij} = 0$, $j = 2,\cdots, i$. So $i = t_{ii}\, f_i$ and $(\phi_i, \phi_i) = (\phi_i, t_{ii}\, f_i) \neq 0$. Contradiction! This shows that

$$f_i = \frac{1}{t_{ii}} (\phi_i - \sum_{j=2}^{i} t_{ij}\, f_j), \qquad i = 2,\cdots,n \;.$$

If $\phi_i \in D(X')$, then $f_i \in D(X')$ which is false. Therefore $\phi_i \in D(X)$ for $i = 1,\cdots,n$.

By means of Th. 1.22, we get a corresponding statement of the previous theorem in terms of matrix representations of c.l.g.H.o.

<u>Theorem 4.2</u>. If X is a c.l.g.H.o., then, for any given positive integer n, there exists a d.s. (ϕ_n) of X such that the q.m. A, which belongs to X for (ϕ_n), and its adjoint A^*, have at least n equal corresponding rows. If X is proper, then, for any given positive integer n, there exists a d.s. (ϕ_n) of X such that the q.m. A, which belongs to X for (ϕ_n), and its adjoint A^*, have at least n unequal corresponding rows.

<u>Theorem 4.3</u>. There exists a c.l.g.H.o. X for which it is possible to choose a d.s. (ϕ_n) of X such that the matrix A, which belongs to X for (ϕ_n), and its adjoint A^*, have no equal corresponding rows.

Proof. Consider the least c.l.e. of the operator Y

given in the example on p.20 . Let it be denoted here by Y for simplicity. Since Y' is c.l., Y' has a d.s. (ψ_n). Let f still be the e.e. given there, and (ψ_{mn}) be the subsequence of (ψ_n) such that $(f, \psi_{mn}) = 0$, $n = 1,2,\cdots$. Define $g = f + \sum_{n=1}^{\infty} a_n \psi_{mn}$, where $a_n = \frac{1}{n}$ if $||Y\psi_{mn}|| \leq 1$, and $a_n = \frac{1}{n||Y\psi_{mn}||}$ if $||Y\psi_{mn}|| > 1$. First, we see that g is in S, for $\sum_{n=1}^{\infty} |a_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Secondly $g \in D(Y)$, for $|| \sum_{n=1}^{\infty} a_n Y \psi_{mn} ||^2 \leq \sum_{n=1}^{\infty} |a_n|^2 < \infty$. On the otherhand $g \neq 0$ and $g \in D(Y')$, otherwise $f \in D(Y')$. Moreover $(g, \psi_{mn}) = a_n \neq 0$, $n = 1,2,\ldots$. Now examine the sequence $g, \psi_1, \psi_2, \cdots$: (1) if Y_1 is the contraction of Y over this sequence, then $\tilde{Y}_1 = Y$; (2) $g \neq 0$; $(g, \psi_n) \neq 0$, $n = 1,2,\cdots$; $(\psi_i, \psi_i) = 1$, $(\psi_i, \psi_j) = 0$, $i \neq j = 1,2,\cdots$. It is clear that the c.o.N.s. (ϕ_n) obtained by orthonormalization of that sequence is a d.s. of Y, and $\phi_n \in D(Y')$, $n = 1,2,\cdots$, (for the same reason that $\phi_1 \in D(X')$ in the proof of Th. 4.1). Therefore the matrix A belonging to X for (ϕ_n) and its adjoint A^* have no equal corresponding rows.

We know that a q.m. which belongs to a H.o. is Hermitian, no matter what d.s. of the operator has been chosen. But, for g.H.o., the situation is entirely different. From the above discussions, we see that representing matrices of a g.H.o. depend much on the choice of d.s. of the operator, at least as far as their Hermitianness is concerned. They and their adjoints can be "quite" alike, but also can be "quite" different. Sometimes, as is shown in Th. 4.3, the matrix and its adjoint can be entirely different. So, just by examining the "degree" of Hermitianness of a q.m., it is hardly possible to know whether it would generate a g.H.o. for some c.o.n.s.

A few other rather pathological characteristics of the matrix representations of g.H.o. should be mentioned briefly.

(1) It seems unlikely that functions of a g.H.o. and its representing matrices could be related correspondingly. For instance, the adjoint A^* of A which belongs to X has nothing to do with X^*. A^* may not even be a q.m.

(2) If A and B both represent a g.H.o. X, then they are, in general, not unitarily equivalent,[a] as is the case for some other kinds of operators.[b]

B. A Few Formal Relations

<u>Theorem 4.4</u>. If X is a c.l.g.H.o., and A,B

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[a] A is unitarily equivalent to B, if there exists a u.m. U such that $B = (UA)U^* = U(AU^*)$.

[b] See reference (11), p. 88, Th. 3.1.

are the matrices which belong to X and X^* for the d.s. (ϕ_n) and (τ_n) of X and X^* respectively, then there exist two u.m. $U = (u_{ij})$ and $V = (v_{ij})$ and a H.q.m. C such that the following relations hold:

$$A\,U = U\,C \quad , \tag{1}$$

$$B\,V = V\,C \quad , \tag{2}$$

$$A = (U\,C)U^* \quad , \tag{3}$$

$$B = (V\,C)V^* \quad , \tag{4}$$

$$V = W\,U \quad , \tag{5}$$

where $W = (w_{ij})$ and $w_{ij} = (\tau_i,\ \phi_j)$, $i,j = 1,2,\cdots$.

Proof. Let $C = (c_{ij})$ be the matrix which belongs to X' for the d.s. (ψ_n) of X'. Define $u_{ij} = (\phi_i,\ \psi_j)$, $v_{ij} = (\tau_i,\ \psi_j)$, $i,j, = 1,2,\cdots$. Then $U = (u_{ij})$, $V = (v_{ij})$ are unitary, and C is a H.q.m. We will show that $A\,U = U\,C$ for illustration. Others can be proved by using the same manipulation.

$$\sum_{k=1}^{\infty} a_{ik}\,u_{kj} = \sum_{k=1}^{\infty} (X\,\phi_i,\ \phi_k)(\phi_k,\ \psi_j)$$

$$= (X\,\phi_i,\ \psi_j) = (\phi_i,\ X\,\psi_j)$$

$$= \sum_{k=1}^{\infty} (\phi_i,\ \psi_k)(\psi_k,\ X\,\psi_j)$$

$$= \sum_{k=1}^{\infty} (\phi_i, \psi_k)(X\, \psi_k, \psi_j) = \sum_{k=1}^{\infty} U_{ik}\, C_{kj}\ ,\ i,j, = 1,2,\cdots .$$

Therefore $A\,U = U\,C$.

Theorem 4.5. If X is a c.l.*-o., and A, B are the matrices which belong to X and X^* for the d.s. (ϕ_n) and (τ_n) respectively, then X is a g.H.o. if and only if there exist two u.m. U, V and a H.q.m. C such that the five matrices A, B, C, U, and V satisfy the relations (1), (2), and (5) given in the previous theorem.

Proof. We proved the necessity in Th. 4.4. To show the sufficiency, let us define:

$$\psi_j = \sum_{k=1}^{\infty} \bar{U}_{kj}\, \phi_k\ ,$$

$$Y\, \psi_j = \sum_{k=1}^{\infty} C_{jk}\, \psi_k\ , \qquad j = 1,2,\cdots .$$

Then
$$(X\, \phi_i, \psi_j) = \sum_{k=1}^{\infty} a_{ik}\, U_{jk}\ ,$$

$$(\phi_i, Y\, \psi_j) = \sum_{k=1}^{\infty} U_{ik}\, C_{kj}\ , \quad i,j, = 1,2,\cdots .$$

Since $A\,U = U\,C$, $(X\,\phi_i, \psi_j) = (\phi_i, Y\,\psi_j)$. So $\psi_j \in D(X^*)$ and $X^*\,\psi_j = Y\,\psi_j$, $j = 1,2,\cdots$. Hence $Y \leq X^*$.

Similarly, if we define

$$\sigma_j = \sum_{k=1}^{\infty} \bar{v}_{kj}\, \tau_k\ ,$$

$$Z\sigma_j = \sum_{k=1}^{\infty} c_{jk}\sigma_k, \; j = 1,2,\cdots,$$

then $Z \leq X^{**} = X$.

We will show that $Y = Z$ and is a H.o.

$$(\psi_j, \tau_k) = \left(\sum_{\ell=1}^{\infty} \bar{U}_{lj}\phi_l, \tau_k\right) = \sum_{\ell=1}^{\infty} \bar{U}_{lj}\bar{W}_{kl}$$

$$(\sigma_j, \tau_k) = \left(\sum_{\ell=1}^{\infty} \bar{v}_{lj}\tau_l, \tau_k\right) = \bar{v}_{kj} = \sum_{\ell=1}^{\infty} \bar{w}_{kl}\bar{u}_{lj},$$

so $(\psi_j, \tau_k) = (\sigma_j, \tau_k)$ $j,k = 1,2,\cdots$. Hence $\psi_j = \sigma_j$, $j = 1,2,\cdots$, and $Y = Z$. But, since $C = (c_{ij})$ is a H.q.m., $Y = Z$ is a H.o.

Now $Y \leq X$, $Y \leq X^*$ and Y is Hermitian. By Lemma 2.1, X is a g.H.o.

C. G.H.O. - generating Matrices

As is pointed out in Section A, just by examining its "degree" of Hermitianness, it is hardly possible to know whether a q.m. could generate g.H.o. for some c.o.N.s. We give, however, some answers to the question in other forms. Unfortunately, most of them are more or less impractical.

<u>Definition 4.1</u>. A q.m. is called a g.H.o.-generating matrix if it generates a c.l.g.H.o. for some c.o.N.s.

<u>Theorem 4.6</u>. A q.m. A is a g.H.o.-generating matrix

if and only if there exist a u.m. U and a H.q.m. C such that $A\,U = U\,C$.

Proof. The necessity was shown in Th. 4.4. To prove its sufficiency, take any c.o.N.s. (ϕ_n). Define

$$\psi_j = \sum_{m=1}^{\infty} \overline{U}_{mj}\,\phi_m ,$$

$$X\,\psi_j = \sum_{m=1}^{\infty} c_{jm}\,\psi_m ,$$

$$X\,\phi_j = \sum_{m=1}^{\infty} a_{jm}\,\phi_m , \qquad \text{for } j = 1,2,\cdots .$$

In case that $\phi_i = \psi_j$, then

$$U_{im} = \begin{cases} 1 & \text{for } m = j , \\ 0 & \text{for } m \neq j . \end{cases}$$

Hence $(X\,\phi_i, \psi_k) = (\sum_{m=1}^{\infty} a_{im}\,\phi_m, \psi_k) = \sum_{m=1}^{\infty} a_{im}\,U_{mk}$,

$$(X\,\psi_j, \psi_k) = (\sum_{m=1}^{\infty} c_{jm}\,\psi_m, \psi_k) = c_{jk}$$

$$= \sum_{m=1}^{\infty} u_{im}\,c_{mk} , \quad k = 1,2,\cdots .$$

Since $A\,U = U\,C$, $(X\,\phi_i, \psi_k) = (X\,\psi_j, \psi_k)$, $k = 1,2,\cdots$, so $X\,\phi_i = X\,\psi_j$. This means: if $\phi_i = \psi_j$, then the two ways of defining X on this element are consistent.

It is easy to show that X' exists and is an extension of the contraction of X over (ψ_n). For:

$$(X\,\phi_i,\ \psi_j) = \sum_{k=1}^{\infty} a_{ik}\,U_{kj} = \sum_{k=1}^{\infty} U_{ik}\,C_{kj} = (\phi_i,\ X\,\psi_j)\ ,$$

and

$$(X\,\psi_i,\ \psi_j) = C_{ij} = \overline{C}_{ji} = (\psi_i,\ X\,\psi_j)\ ,\ i,j = 1,2,\cdots\ .$$

Now X is the c.l.g.H.o. which is generated by A for (ϕ_n).

So far we have been concerned with matrices only. There are,however, corresponding statements which can be made for operators.

<u>Theorem 4.7</u>. If X is a c.l.*-o., and A the matrix which belongs to X for the d.s. (ϕ_n), then X is a g.H.o. if and only if there exist a H.g.m. C and a u.m. U such that $A\,U = U\,C$. In this case X' is an extension of the c.l.H.o. Z which is generated by C for the c.o.N.s. (ψ_n), where $\psi_j = \sum_{m=1}^{\infty} \overline{U}_{mj}\,\phi_m$, $j = 1,2,\cdots$.

Proof. We showed the necessity in Th. 4.4. It was also shown that if $A\,U = U\,C$, then A generates a c.l.g.H.o. for any c.o.N.s. (Th. 4.6). So A generates a c.l.g.H.o. Y for the d.s. (ϕ_n) of X. Since X and Y have the same elementary operator, they themselves must be the same. Finally $X' = Y' \geq Z$.

In Th. 4.4, we proved a few relations necessarily satisfied by q.m. A, B which generate g.H.o. Later it was shown that relations (1) and (2) are each sufficient for the matrices A, B to generate g.H.o. (Th. 4.6). It is natural to expect that relations (3) and (4) possess the same sufficiency. In fact, there is "evidence" for that: for since $A\,U = U\,C$ is sufficient for A to generate g.H.o., then if the associative law held, $A = (U\,C)U^*$ would also be sufficient. However, we need a proof.

<u>Theorem 4.8</u>. If A is a q.m., then A is a g.H.o.-generating matrix if and only if there exist a H.q.m. C and u.m. U such that:

(1) $U\,C$ is a q.m.,

(2) $A = (U\,C)U^*$.

A will be Hermitian if and only if $(U\,C)U^* = U(C\,U^*)$.

Proof. In Th. 4.4, we showed that if A is a g.H.o.-generating matrix, then there exists a H.q.m. C and a u.m. U such that $A = (U\,C)U^*$. Further, we know that

$$\sum_{j=1}^{\infty} \Big| \sum_{k=1}^{\infty} u_{ik}\, c_{kj} \Big|^2 = \sum_{j=1}^{\infty} |(X\,\phi_i, \psi_j)|^2 = \|X\,\phi_i\|^2 < \infty\,, \qquad i = 1, 2, \cdots .$$

So $U\,C$ is a q.m.

To show that the conditions are sufficient, let us take a c.o.N.s. (ψ_n) and define

$$X\,\psi_j = \sum_{m=1}^{\infty} c_{jm}\,\psi_m\,,$$

$$\phi_j = \sum_{m=1}^{\infty} U_{jm}\,\psi_m\,,$$

$$X\,\phi_j = \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} u_{jm}\,c_{mn}\Big)\psi_n\,,\quad j = 1,2,\cdots\,.$$

It can be seen that if $\phi_i = \psi_j$, then $X\,\phi_i = X\,\psi_j$, so X is consistently defined. We can also see that $(X\,\phi_i,\,\psi_j) = (\phi_i,\,X\,\psi_j)$, and $(X\,\psi_i,\,\psi_j) = (\psi_i,\,X\,\psi_j)$, $i,j = 1,2,\cdots$. Therefore X is a g.H.o. and so is $\widetilde{X}$. But A is a q.m.[a] and $\widetilde{X}$ is generated by A for (ϕ_n), for

$$(X\,\phi_i,\,\phi_j) = \Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{im}\,c_{mn}\,\psi_n,\,\phi_j\Big)$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{im}\,c_{mn}\,u^{*}_{nj} = a_{ij}\,,\quad i,j = 1,2,\cdots\,.$$

Finally A is Hermitian if and only if $A = A^{*}$. But $A = U(C\,U^{*})$. Hence A is Hermitian if and only if $(U\,C)U^{*} = U(C\,U^{*})$.

The previous theorem can be stated in a different form with emphasis on a given H.q.m.

<u>Theorem 4.9</u>. If C is a H.q.m., then for each u.m. U $(U\,C)U^{*}$ will be a g.H.o.-generating matrix, provided that $U\,C$ is a q.m. $(U\,C)U^{*}$ is Hermitian if and only if $(U\,C)U^{*} = U(C\,U^{*})$.

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[a] To prove that A is a q.m., see reference (6), p. 211, foot note 7.

Another theorem of the same kind is the following.

<u>Theorem 4.10</u>. If A is a q.m., then A is a g.H.o.-generating matrix if and only if there exists a u.m. U such that:

(1) $A\ U$ and $(A\ U)^*$ are q.m.,

(2) $U^*(A\ U)$ is a H.q.m.

Proof. The conditions are necessary. For, by Th. 4.4:

$$\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{ik} u_{kj} \right|^2 = \sum_{j=1}^{\infty} |(X\,\phi_i, \psi_j)|^2 = \|X\,\phi_i\|^2 < \infty ,$$

$$\sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{ik} u_{kj} \right|^2 = \sum_{i=1}^{\infty} |(X\,\psi_j, \phi_i)|^2 = \|X\,\psi_j\|^2 < \infty ,$$

so both $A\ U$ and $(A\ U)^*$ are q.m. Moreover,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u^*_{im}\, a_{mn}\, u_{nj} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\psi_i, \phi_m)(X\,\phi_m, \phi_n)(\phi_n, \psi_j)$$

$$= \sum_{m=1}^{\infty} (\psi_i, \phi_m)(X\,\phi_m, \psi_j)$$

$$= \sum_{m=1}^{\infty} (\psi_i, \phi_m)(\phi_m, X\,\psi_j) = (\psi_i, X\,\psi_j) ;$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u^*_{im}\, a^*_{mn}\, u_{nj} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\psi_i, \phi_m)(\phi_m, X\,\phi_n)(\phi_n, \psi_j)$$

$$= \sum_{n=1}^{\infty} (\psi_i, X\,\phi_n)(\phi_n, \psi_j)$$

$$= \sum_{n=1}^{\infty} (X\,\psi_i, \phi_n)(\phi_n, \psi_j) = (X\,\psi_i, \psi_j) .$$

Hence $U^*(A\ U)$ is a H.q.m.[a]

To show its sufficiency, let (ϕ_n) be a c.o.N.s. Define

$$\psi_j = \sum_{m=1}^{\infty} \bar{u}_{mj}\, \phi_m ,$$

$$X\, \phi_j = \sum_{m=1}^{\infty} a_{jm}\, \phi_m ,$$

$$X\, \psi_j = \sum_{n=1}^{\infty} \Big(\sum_{m=1}^{\infty} u^*_{jm}\, a^*_{mn} \Big)\, \phi_n \ , \ j = 1,2,\cdots .$$

It can be shown that X is consistently defined, and $(X\,\phi_i,\ \psi_j) = (\phi_i,\ X\,\psi_j)$, $(X\,\psi_i,\ \psi_j) = (\psi_i,\ X\,\psi_j)$, $i,j = 1,2,\cdots$. Hence $X^{\dagger}$ exists. Thus $\tilde{X}$ is the c.l.g.H.o. generated by A for (ϕ_n).

Corresponding statements for operators can be made without difficulty. For instance:

<u>Theorem 4.11</u>. If X is a c.l.*- o., and A is the matrix which belongs to X for the d.s. (ϕ_n), then X is a g.H.o. if and only if there exist a H.q.m. C and a u.m. U such that:

(1) $U\,C$ is a q.m.,

(2) $A = (U\,C)U^*$.

In this case, $X^{\dagger}$ is an extension of the c.l.H.o. generated by C for the c.o.N.s. (ψ_n), where $\psi_j = \sum_{m=1}^{\infty} \bar{u}_{mn}\, \phi_m$, $j = 1,2,\cdots$. Further X is Hermitian if and only if $(U\,C)U^* = U(C\,U^*)$.

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[a] See Th. 4.9, foot note.

Another thing we should mention here is: if X is a c.l.*-o., then X is a g.H.o. if and only if X^* is so, hence all theorems concerning the matrix A belonging to X for a d.s. of X remain true if A is replaced by a matrix B which belong to X^* for a d.s. of X^*.

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